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On Finite Approximation

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1 Introduction

In [1] we constructed a quantum 2-torus and studied its model theoretic properties. Next step would be the study of definable bijections between $T_q^2(\mathbb{F})$ and $T_{q'}^2(\mathbb{F})$, analogue of regular isomorphisms between algebraic varieties in algebraic geometry.

Recall three main theorems proved in [1] ;

(1) $\mathcal{L}_{\omega_1,\omega}$-theory of the quantum 2-torus is $\aleph_1$-categorical.

(2) The theory of quantum line-bundles is superstable.

(3) With the pairing function, within $(\Gamma, \cdot, 1, q)$ we can define $(\Gamma, \oplus, \otimes, 1, q)$, and

$$(\Gamma, \oplus, \otimes, 1, q) \simeq (\mathbb{Z}, +, \cdot, 0, 1).$$

Hence the theory of the quantum 2-torus $(\mathbf{U}, \mathbf{V}, \mathbb{F}^*, \Gamma)$ with the pairing function is undecidable and unstable.

In [2] we associate quantum 2-tori T_θ with the structure over

$$\mathbb{C}_\theta = (\mathbb{C}, +, \cdot, y = x^\theta),$$

where $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and introduce the notion of geometric isomorphisms between such quantum 2-tori.

We showed that the notion of geometric isomorphisms is closely connected with the fundamental notion of Morita equivalence of non-commutative geometry.

*joint work with Boris Zilber, Oxford University

Theorem 1 *The quantum 2-tori T_{θ_1} and T_{θ_2} are Morita equivalent if and only if*
$$\theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d} \textit{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \textit{ with } |ad - bc| = 1.$$

Having Theorem 1, we study the isomorphism type of T_θ with respect to $\mathrm{GL}(2, \mathbb{Z})$. For this we consider the structure $\overline{\mathbb{R}}/E$ where E is the equivalence relation defined by

$$\theta_1 E \theta_2 \iff \left(\theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}),\ |ad - bc| = 1\right).$$

In the next section we introduce the notion of finite approximation and weak ring defined by Zilber in [4] and [5]. Then we study the equivalence relation from a finite approximation point of view. In section 3 we introduce the notion of finitely approximated subset of $\mathbb{R}$ and show that any finitely approximated subset is a closed subset of $\mathbb{R}$.

2 Finite approximation

Let L be a language. Consider the following situation:

$$\begin{array}{ccccccccc} M_1 & \subset & M_2 & \subset & \cdots & M_n & \subset & \cdots & M^* \\ &&&&&&&& \Big\downarrow \mathrm{lim} \\ &&&&&&&& N \end{array}$$

Here each M_i is a finite L-structure. We view an infinite L- structure M^* as a limit of the sequence capturing all the properties of M_i's, e.g., ultraproduct of those finite structures. N is another infinite L-structure. The mapping $\lim : M^* \to N$ is a homomorphism. We are interested in subsets X of N^n that are finitely approximated (defined in section 3) by the sequnce of finite structures.

From now on,

- ${}^*\mathbb{Z}$ is the saturated nonstandard integers,
- μ is a highly divisible infinite nonstandard integer,
- $\overline{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$, i.e, $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$,
- P^4 is a 4-ary relation.

Consider the following situation;

$$\begin{array}{c}({}^*\mathbb{Z}/\mu^2, +, P^4)\\ \Big\downarrow \lim_\mu \\ (\overline{\mathbb{R}}, +, P^4)\end{array}$$

where $\lim_\mu$ is a homomorphism called *finite approximation* defined as follows;

for α with $-\dfrac{\mu^2}{2} \leq \alpha < \dfrac{\mu^2}{2}$ we have

$$\frac{\alpha}{\mu} \in {}^*\mathbb{Q}$$

where ${}^*\mathbb{Q}$ is the nonstandard rational numbers and we set

$$\lim_\mu(\alpha) = \mathrm{st}\left(\frac{\alpha}{\mu}\right).$$

The interpretation of P^4 is defined as follows

- on ${}^*\mathbb{Z}/\mu^2$, $P^4(a_1, b_1, a_2, b_2) \Longleftrightarrow a_1 b_1 = a_2 b_2$,
- on $\overline{\mathbb{R}}$, $P^4(x_1, y_1, x_2, y_2) \Longleftrightarrow$
  $[(x_1 y_1 \equiv x_2 y_2 \mod \mathbb{Z}) \bigvee (x_1 = \infty, y_1 = \infty)]$.

Observe that

$$x_1 y_1 \equiv x_2 y_2 \mod \mathbb{Z} \Longleftrightarrow e^{2\pi i(x_1 y_1)} = e^{2\pi i(x_2 y_2)}$$

and

$$\overline{\mathbb{R}} \times \overline{\mathbb{R}} \xrightarrow{e^{2\pi i(xy)}} S \subset \mathbb{C},$$

where S is a unit circle and viewed as a multiplicative group.

We have the following commutative diagram;

$$\begin{tikzcd}
{}^*\mathbb{Z}/\mu^2 \times {}^*\mathbb{Z}/\mu^2 \arrow[r, "a \cdot b"] \arrow[d] & {}^*\mathbb{Z}/\mu^2 \arrow[d] \\
\overline{\mathbb{R}} \times \overline{\mathbb{R}} \arrow[r, "e^{2\pi i(xy)}"] & S \subset \mathbb{C}
\end{tikzcd}$$

Observe also, $e^{2\pi i \frac{ab}{\mu^2}} \in S$.

2.1 Equivalence relation $E^{^*\mathbb{Z}}$ on $^*\mathbb{Z}/\mu^2$

We define an equivalence relation $E^{^*\mathbb{Z}}$ on $^*\mathbb{Z}/\mu^2$ corresponding to the equivalence relation $E^{\mathbb{R}}$.

Definition 2 **($E^{^*\mathbb{Z}}$)** *Let $\alpha_1, \alpha_2 \in {}^*\mathbb{Z}/\mu^2$ be such that*

$$-k\mu \leq \alpha_1, \alpha_2 < k\mu, \text{ for some } k \ll \mu.$$

Then

$$E^{^*\mathbb{Z}}(\alpha_1, \alpha_2) \overset{\text{def}}{\iff}$$
$$\exists\, a, b, c, d, \beta \in {}^*\mathbb{Z} \Big[(|ad - bc| = 1) \wedge (\mu\beta = \alpha_1\alpha_2) \wedge (a\alpha_1 + b\mu = c\beta + d\alpha_2)\Big].$$

Remark 3 *We want to have the following equation*

$$\frac{a\frac{\alpha_1}{\mu} + b}{c\frac{\alpha_1}{\mu} + d} = \frac{a\alpha_1 + b\mu}{c\alpha_1 + d\mu} = \frac{\alpha_2}{\mu}. \tag{1}$$

By multiplying both sides by μ we get

$$\frac{a\alpha_1\mu + b\mu^2}{c\alpha_1 + d\mu} = \alpha_2,$$

which may look equivalent to

$$(a\alpha_1\mu + b\mu^2 = c\alpha_1\alpha_2 + d\alpha_2\mu). \tag{2}$$

However, in $^*\mathbb{Z}/\mu^2$, we have $b\mu^2 \approx 0$. Thus we introduce β and the relation

$$(\mu\beta = \alpha_1\alpha_2) \wedge (a\alpha_1 + b\mu = c\beta + d\alpha_2)$$

replaces the equation (2) in order to define the equivalence relation $E^{^*\mathbb{Z}}(\alpha_1, \alpha_2)$ above.

The relation $E^{\mathbb{R}}$ has the following property;

Proposition 4 *For any $\theta_1, \theta_2 \in \mathbb{R}$, if $E^{\mathbb{R}}(\theta_1, \theta_2)$ then there are $\alpha_1, \alpha_2 \in {}^*\mathbb{Z}$ such that $E^{^*\mathbb{Z}}(\alpha_1, \alpha_2)$ where $\lim_\mu(\alpha_1) = \theta_1$ and $\lim_\mu(\alpha_2) = \theta_2$.*

Proof: Take $\theta_1, \theta_2 \in \overline{\mathbb{R}}$ such that $E^{\mathbb{R}}(\theta_1, \theta_2)$. Fix $a, b, c, d \in \mathbb{Z}$ such that

$$\theta_2 = \frac{a\theta_1 + b}{c\theta_1 + d}.$$

Take α_1 such that

$$\text{st}\left(\frac{\alpha_1}{\mu}\right) = \theta_1.$$

Claim 5 *There are* $k_1, n_1 \in {}'\mathbb{Z}$ *such that* $k_1, n_1 \ll \mu$ *and*

$$\mathrm{st}\left(\frac{k_1}{n_1}\right) = \mathrm{st}\left(\frac{\alpha_1}{\mu}\right) = \theta_1,$$

where $k_1, n_1 \in {}'\mathbb{Z}$ *means that* k_1, n_1 *are nonstandard integers (but not infinite).*

Proof: Take $k_1, n_1 \in {}'\mathbb{Z}$ to satisfy

$$\frac{ak_1 + bn_1}{ck_1 + dn_1} \quad \approx \quad \frac{a\frac{\alpha_1}{\mu} + b}{c\frac{\alpha_1}{\mu} + d}.$$

QED

Put

$$\frac{k_2}{n_2} = \frac{ak_1 + bn_1}{ck_1 + dn_1},$$

and set $\alpha_2 = \dfrac{\mu k_2}{n_2}$. Then we have $\mathrm{st}\left(\dfrac{\alpha_2}{\mu}\right) = \theta_2$ and

$$\begin{aligned}\frac{a\frac{\alpha_1}{\mu} + b}{c\frac{\alpha_1}{\mu} + d} &= \frac{a\alpha_1 + b\mu}{c\alpha_1 + d\mu} \\ &= \frac{\alpha_2}{\mu}.\end{aligned}$$

Thus

$$\theta_2 = \frac{a\theta_1 + b}{c\theta_2 + d}.$$

Put now

$$\beta := \alpha_1 \frac{k_2}{n_2}.$$

Then we have

$$(\mu\beta = \alpha_1\alpha_2) \wedge (a\alpha_1 + b\mu = c\beta + d\alpha_2.$$

Therefore $E^{^*\mathbb{Z}}(\alpha_1, \alpha_2)$ holds.

3 Finitely approximated subsets of reals

We work in the following situation;

$$\begin{array}{ccc} \Phi & \subset & \left({}^*\mathbb{Z}/\mu^2\right)^n \\ \Big\downarrow \lim_\mu & & \Big\downarrow \lim_\mu \\ \Psi & \subset & \left(\overline{\mathbb{R}}\right)^n \end{array} .$$

Here Φ is a 1st-order definable set and Ψ is a set.

Definition 6 *If Φ, Ψ satisfy the following conditions;*

- *for any $\overline{\alpha} = (\alpha_1, \cdots, \alpha_n)$ with $-\dfrac{\mu^2}{2} \le \alpha_1, \cdots, \alpha_n < \dfrac{\mu^2}{2}$, if $\overline{\alpha} \in \Phi$ then $\lim_\mu(\overline{\alpha}) \in \Psi$,*
- *for any $\overline{\theta} = (\theta_1, \cdots, \theta_n)$ with $\theta_i \in \overline{\mathbb{R}}$, if $\overline{\theta} \in \Psi$, then there are $\alpha_1, \cdots, \alpha_n$ with $-\dfrac{\mu^2}{2} \le \alpha_1, \cdots, \alpha_n < \dfrac{\mu^2}{2}$, such that $\overline{\alpha} \in \Phi$ and $\lim_\mu(\overline{\alpha}) = \overline{\theta}$,*

then we say that Ψ is finitely approximated by Φ.

Proposition 7 *Suppose Φ is a 1st-order definable set such that Ψ is finitely approximated by Φ. Then Ψ is closed in the usual metric topology. In other words for a set of $\left(\overline{\mathbb{R}}\right)^n$ being closed is a necessary condition for being finitely approximated.*

Proof:

We work in one-dimensional case. The other cases are similar. Proof is by contradiction. Assume Ψ is not closed.

Consider an infinite sequence $\theta_1, \theta_2, \cdots, \theta_n, \cdots \in \overline{\mathbb{R}}$ such that for each n, $\theta_n \in \Psi$, $\lim\limits_{n\to\infty} \theta_n = \theta$ and $\theta \notin \Psi$.

Since Ψ is finitely approximated there is an infinite sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots \in {}^*\mathbb{Z}/\mu^2$ such that for each n, $\alpha_n \in \Phi$, and $\lim_\mu(\alpha_n) = \theta_n$.

Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n, \cdots$ be an infinite sequence of rational numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. Since $\lim\limits_{n\to\infty} \theta_n = \theta$, we may assume each ε_n satisfy

$$\frac{1}{2}|\theta - \theta_n| = \varepsilon_n.$$

Then there is an N such that for any $n > N$ we have

$$|\theta_N - \theta_n| < \varepsilon_n.$$

Now consider a type

$$t(x) = \{x \; : \; |x - \alpha_n| < \varepsilon_n \, \mu\}.$$

It is easy to see that $t(x)$ is satisfiable. By saturation there is an $\alpha \in {}^*\mathbb{Z}/\mu^2$ such that α realizes $t(x)$ and $\alpha \in \Phi$. Further we have

$$|\alpha - \alpha_n| < \varepsilon_n \, \mu \quad \text{for all } n.$$

Set $\theta' := \lim_\mu(\alpha) = \mathrm{st}\left(\frac{\alpha}{\mu}\right)$. Then for each n we have $|\theta' - \theta_n| < \varepsilon_n$. Thus $\theta' = \theta$. Now we have a contradiction since $\theta \notin \Psi$ and $\theta' \in \Psi$. **QED**

3.1 A sufficient condition for finite approximation

We examine carefully the argument in the proof of Proposition 7.

Let Ψ be a subset of $\overline{\mathbb{R}}$, and $\{\theta_n : n \in \mathbb{N}\}$ be a sequence in Ψ such that $\theta = \mathrm{lin}_{n\to\infty}$ in the usual metric topology.

For a 1st-order definable set Φ to finitely approximate the subset Ψ, we need the following;

For any infinite sequence $\{\alpha_n : n \in \mathbb{N}\}$ in Φ with

- for each n, $\lim_\mu(\alpha_n) = \theta_n$,
- α realizes the type $t(x)$ defined in the same way as in the proof of Proposition 7.

then $\alpha \in \Phi$ and $\lim_\mu(\alpha) = \theta$.

$$\begin{tikzcd}
\Phi \arrow[d, "\lim_\mu"] & \alpha_1 & \alpha_2 & \cdots & \alpha_n & \cdots & \alpha & {}^*\mathbb{Z}/\mu^2 \arrow[d, "\lim_\mu"] \\
\Psi & \theta_1 & \theta_2 & \cdots & \theta_n & \cdots & \theta & \overline{\mathbb{R}}
\end{tikzcd} .$$

If this is the case we say that the metric topology with respect to Ψ and the topology in ${}^*\mathbb{Z}/\mu^2$ with respect to Φ coincide.

In summary we have

Proposition 8 *Closed set Ψ is finitely approximated by Φ if and only if the metric topology with respect to Ψ and the topology in ${}^*\mathbb{Z}/\mu^2$ with respect to Φ coincide.*

This gives us a sufficient condition for finite approximation.

References

[1] Masanori Itai, Boris Zilber, *Notes on a model theory of a quantum 2-torus T_q^2 for generic q*, arXive:1503.06045v1, 2015

[2] Masanori Itai, Boris Zilber, *A model theoretic Rieffel's theorem of quantum 2-torus* arXiv:1708.02615v1, 2017

[3] Masanori Itai and Boris Zilber, *On finite approximation*, in preparation, 2019

[4] Boris Zilber, *The semantics of the canonical commutation relation* arxiv:1604.07745v1, 2016

[5] Boris Zilber, *Structural approximation and quantum mechanics (Draft)*, preprint, 2018